The Power of 2

Frank Tapson

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Then it was entitled *Doubling Up*.
The original article has been modified slightly.

Take a piece of paper – you may use any size you please – and fold it in half, then fold it in half again, and yet again … How many times do you think you can do this? If you have never met this problem before then try it before you read any further – you will very probably receive a surprise. Have a guess first before actually trying to do the folding and then see how far you get.

Many people on meeting this problem for the first time are prepared to say that, provided the paper is large enough then it may be folded in half any number of times. Well, as you might have discovered by now, after 7 such foldings the task becomes extremely difficult, and if not impossible then it will almost certainly be so after the next fold. It is interesting to consider what is happening.

After the first fold, the piece of paper we have to work on next is double the thickness of the original. Another fold of this piece doubles that thickness so that we now have 4× the thickness of the original. Folding again will once more double-up on the thickness so that we have (2×2×2) 8 thicknesses of paper. This is followed by 16× after the next fold, then 32, then 64, and 128 after the 7th fold. Even the thinnest paper, after 7 folds could have a thickness of about 5 mm.

Now that thickness of paper could certainly be folded in half usually, but there is an added difficulty. Just as the thickness has been doubled with each fold so the area has been halved, and after only 6 foldings we are usually trying to bend something which is not much bigger than an extra-large postage stamp, which is why that piece of paper is so difficult to fold.

It is interesting to wonder how far the process might be taken if a piece of super-large paper were used. Let us assume it still very thin, but that we can start with a piece the size of a football-pitch. Go on - have a guess, how many times would you manage to fold it in half?

Some might wish to argue about the precise stage at which the task becomes impossible, but if the 13th fold could be made, it produces something which is about one metre square and 400 mm (nearly half a metre) thick. Think about bending that!

So let's forget folding and change it to cutting in half, piling the cut pieces together, cutting the pile in half and so on. It is the same problem, just a little bit more credible. (But only a very little bit as we shall see.)

Now to carry on with our speculation about how thick, or how high, the pile might eventually be. Taking a big leap forward we can ask, just how high would the resulting pile be after 100 cuts, and could we stand on top of it? It is impossible of course, but fun to see why.

By now you must some idea of what to expect – or have you? After the 25th cut we would have a pile which would be over a kilometre high. The 53rd cut would produce a pile reaching up to touch the sun. The 83rd cut make a piles whose top is somewhere the centre of our galaxy, from which it follows that the pile produced after the 84th cut is reaching the other side of our galaxy. And there we will let the matter rest.

Of course it needs to be remembered that all this is totally impossible in practice if only because while the pile doubles in height after every cut, it is also halving in its cross-sectional area. And no matter what its original size was, by the time we had got to the 50th cut we would be trying to pile molecules one on top of the other!

No matter, this simple concept of the growth of the doubling sequence has had a fascination for those concerned with the lighter side of mathematics for many years. Perhaps the most famous is the story told around invention of the chess-board, how the king was so pleased that he offered the inventor any reward that he cared to name. This the inventor did in the form 'one grain of corn on the first square of the board, two grains on the second square, four grains on the third square and so on …' The king thought this a very light price to pay for such a great game and readily agreed. However, he was not at all pleased to learn that the total quantity of grain required could not be supplied by the entire world output of grain for several years to come. Some accounts of the story go on to claim that he had the inventor beheaded for imposing such a mathematical joke upon royalty! Re-telling this story in his major work *A History of Chess* H J R Murray says that the quantity of grain needed is such as to cover England to a depth of 12 metres.

Another form of the story involves either the sale of a horse, or the shoeing of one. In either case the price is fixed at one farthing (over a 100 years ago), or a penny, for the first nail in its shoes, doubled-up for the second nail, doubled again for the third nail and so on. It soon becomes serious money. The only variations in the telling of the story concern the total number of nails (like whether there are 6, 7 or 8 nails per shoe).

A story can also be woven around the telling of a secret to two friends, each of whom tells it to two other (different) friends, each of whom … Assuming that the actual telling occupies just one minute, and that another minute is lost in scurrying off to find someone else to tell, how many people would know after one hour? By now it will come as no surprise to learn that the figure is about one-third of the world population.

Another place where the doubling sequence makes its appearance is in the Tower of Hanoi problem. There the story has monks moving pieces at the rate of one every second. There are 64 pieces to be moved and, when the last one is in place the world will disappear in a clap of thunder. Since that means $2^{64} - 1$ moves have to be made there is no immediate threat!

Reflect on some of this next time you fold a piece of paper in half!