A

Mathematics

Formulary

Study Guide

This Guide is intended for use with the book.
Its use without that will be limited.

The work contained herein has various purposes which can broadly be described as

- gaining familiarity with the book
- learning how to find and use formulas
- exploring numbers and their relationships

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A Mathematics Formulary Study Guide (SG) ~ Notes ~ 1

These notes are written in support of the various worksheets which follow, providing an overview of what is available and some suggestions as to how they might be used.

Familiarisation (SG ~ 1)

In order to derive the greatest benefit from any work of reference the user needs to have some familiarity with it. The sole purpose of this sheet is to provide that familiarity. The work involved is mainly a matter of finding the page, and then copying out the specific piece of information asked for. Some help is provided in the earlier questions by highlighting the words which need to be found in the Contents List to locate the correct page. After working through this sheet, any user should have a very good idea of what the book contains and where (or how) it is to be found.

Conversions (SG ~ 2)

Currently we live in a world that is in the course of changing from a mixture of widely-varying systems of units to one universal coherent system (the **SI System** or, le Système International d'Unités, as it is known in full) and that means we have to be able to change measures given in one sort of units into their equivalent values using another sort of units. The book contains much of the necessary information needed to do that, and this sheet provides some practice in finding and using the information.

Tests A and B *see later note for details on all tests*

The ability to be able to use a formula is an important practical skill in many professions, and the following sheets are intended to help develop that skill by beginning with some explanation of what is happening, and then by using examples from the book.

All the sheets have work to be done. This is in two forms:

- **CheckOuts** where users can make sure they have understood the preceding explanation. Answers are given in a random order at the foot of the sheet.
- **Exercises** where answers are not immediately available. But, as an additional check on correctness of working, in the early stages (Exercises 4 to 7) all answers are **exact**. A calculator answer with a full display of digits after a decimal point must be wrong.

Order (SG ~ 3)

This is an introduction to the idea that, just as in ordinary language we have a need for a grammatical structure, so also in mathematics we must have some rules so that any mathematical statement (such as a formula) will mean the same thing to all who read or use it. This sheet uses numbers only (no letters yet) to illustrate how the order in which operations are done can produce different answers. Some rules are given, including the first concerning the idea of precedence of some operations over others. This latter idea is the dominant thread in all that follows.

(A demonstration that simple left to right working is not possible in all cases has not been offered, but a look at something like Example 6 should convey some idea of the problem.)

Brackets (SG ~ 4)

These are not operators, but signs that show how one group of things must be treated before others. That is, they have the highest precedence throughout this work, and the need to see (and deal with) that is of paramount importance.

A Mathematics Formulary Study Guide (SG) ~ Notes ~ 2

Lines (SG ~ 5)

These make their appearance in three different ways and each with a different meaning. Their kinship with brackets is made clear.

Tests C and D *see later note for details on all tests*

Calculators
Whilst it is expected that calculators will be used throughout this work, considerable care needs to be taken. A basic 4-function model means that all the rules will have to be followed. However, a more sophisticated model (where brackets can be inserted) will deal with some of the exercises just by keying them in as they are written. The square-root key always calls for some care in its use. Understanding of, and familiarity with, one particular model, supported by regular practice, is the best way forward.

From here on, while the explanations can be followed, the Exercises cannot be done without access to the book *A Mathematics Formulary*.

Letters (SG ~ 6)

How letters are used to represent numbers.
No units are used or mentioned at this stage. The focus is solely on finding and using formulas which require the explanations and rules given so far. After finding the correct shape it is necessary to pick the correct formula, when several are on offer, making sure it contains the letters given in the question.
Exercise 7A requires π to be used. Now, most of the answers do not work out exactly. No accuracy has been specified for the answers. It is left for the teacher to indicate what should be done. On the lines of "3 or 4 decimal places" or "3 or 4 significant figures" or "use your own judgement" or "same number of significant figures as the largest value in the given data" or **...**

Contractions (SG ~ 7)

The disappearing multiplication sign.
All remarks from the previous paragraph still apply. It is reasonable to assume that users at this level will have met indices before, and the one-line reference to them is merely a reminder to show how they fit into the idea of a contraction. Only positive integer indices are used and they are the ones most commonly met with in working with formulas. Fractional and negative indices come much later.
Just how solutions are to be written out will need some thought by, and direction from, the teacher. It is easier to find mistakes if something like the format of Example 8 is followed (especially in early work) but everyone soon develops their own shortcuts.
For many, it will be necessary to record intermediate values anyway as using the calculator becomes increasingly demanding.
After gaining some familiarity with the ideas on this sheet it is instructive to look at the various styles used in the *Formulary* to see where simplifications have been made and where they haven't, and how, certainly in the earlier pages, formulas have been made as explicit as possible.

Summary (SG ~ 8)

A one-sheet summary (of sheets SG ~ 3 to SG ~ 7) of the rules and conventions governing the use of formulas, with a few additional notes and reminders. But now they are placed in their order of priority rather than the order in which they were encountered in sheets SG 3 to 7 – which was a teaching order. It was that final order which gave rise to the famous BODMAS which can raise more questions than it answers.
Here the equivalent would be the unpronounceable BLCM. If it is to be made memorable it needs a mnemonic in the form of a sentence. Try "Best Look Cool Man!" or invent your own.

A Mathematics Formulary Study Guide (SG) ~ Notes ~ 3

Summary (SG ~ 8)

A one-sheet summary (of sheets SG ~ 3 to SG ~ 7) of the rules and conventions governing the use of formulas, with a few additional notes and reminders. But now they are placed in their order of priority rather than the order in which they were encountered in sheets SG 3 to 7 – which was a teaching order. It was that final order which gave rise to the famous BODMAS which can raise more questions than it answers.
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Formulas for Mensuration (SG ~ 9)

Using the principles covered on the previous pages in the context of problems.
Units are introduced, as well as the need to match words to the letters used in the formulas.
Making sure the units of the answer match what is being found might need reinforcement. For example, if the lengths are given and a volume is being found, then the answer must be in a cubic version of the units of length: cm must lead to cubic cm (cu.cm or cm^3 as preferred).
In Example 9, the reference to the given formula having a slightly shorter form (in fact, it is the one most likely to be found in general use) could be a very good starting point for a discussion on how many of the given formulas have a shorter form, and even why they have been made so explicit.
In **Exercise 9** the last two, (j) and (k), might be a little more difficult. The former involves a change of units, the latter a cube root.
Further exercises on mensuration can be found from the *TRoL* menu under *Exercises and Practice*, but they will need looking at carefully, bearing in mind they were all produced for a different purpose.
Calculator Exercises offers much that would be suitable. Sheets 2 to 7, 10 and 11, are the best.
Mensuration has work on the square, circle, sector, cube, cylinder, and sphere.

A scientific calculator will be required, in many places, to get trigonometric ratios.

Formulas for Trigonometry (SG ~ 10)

This involves only two pages of *A Mathematics Formulary*:
Exercise 10. The questions offered cover the six different types of problem usually associated with the right-angled triangle. It is assumed that anyone doing this work will be familiar with finding an angle from a trigonometric ratio and vice-versa. The real purpose of having such a table (of formulas) is to help those who are often uncertain about which formula to use. It is all too common to see pupils who can write out the basic formula (from SOHCAHTOA), but then re-order it incorrectly when trying evaluate the component asked for.
Exercise 11. Much the harder one. Qn.1 is structured to give help in finding the correct formulas and and also in gaining a more detailed idea of what is on the page and where it is to be found. Qn.12 is the ambiguous case.
Further suitable exercises can be found from the *TRoL* menu under *Exercises and Practice*, but they will need looking at carefully, bearing in mind they were all produced for a different purpose.
Trigonometry is an obvious place to look. Sheets 6, 7, 15, 21, 27, 28, 31 to 37, are probably the best.
Calculator exercises is a less obvious place to look but also contains a wide range of problems.

Formulas for Poly-shapes (SG ~ 11)

Here the required formula is not immediately on offer, but has to be assembled by using the data provided in the table.
These are often needed when working with polygons and polyhedrons. For instance, finding the radius of the circumcircle that has to be drawn in order to construct a polygon with a specific length of edge. The more difficult task of constructing a polygon (or polyhedron) with a specific area (or volume) has not been asked as it requires the ability to manipulate a formula. However, if it is suitable, then it is easy to make up a few questions on that theme.
In **Exercise 12** the last four questions are not quite so straight-forward as the others.

A Mathematics Formulary Study Guide (SG) ~ Notes ~ 4

Tests E and F *see later note for details on all tests*

Formulas for Professionals (SG ~ 12)

Note that sheet SG~12A is needed to do the work given here.
This is intended to give some idea of the variety of formulas available to, and used by, those doing many different types of work.

Formula Cross Puzzle (SG ~ 13)

Further practice in a different form. While much of the puzzle can be done without it, the *Mathematics Formulary* would be a great help for most pupils.

Tests

The tests, and their answers, are contained in the Answers document (*see later note*).
All the tests exist in two parallel versions. This allows them to be used in a conventional classroom situation where those taking the test are sitting side by side. If each is doing a different (but parallel) test simple copying is not possible, and almost any other form of cheating should be easily observed.
Tests A, B cover the work on sheets SG1 and 2.
Tests C, D cover the work on sheets SG3, 4, and 5. Only in these tests is the *Formulary* not needed.
Also, the use of a scientific calculator makes these tests very easy. As a check on working, ALL the questions have exact answers though this is only really of help in the last section.
Tests E, F cover the work on sheets SG9, 10, and 11.
It is not anticipated that all the tests would be done and in some circumstances instructions on questions to be left out might well be given.

The foregoing work is concerned with the practical (utilitarian) use of mathematics. That which follows deals with the most basic ingredient, the numbers needed to enable mathematics to be explored and their many relationships. This is the beginnings of the topic sometimes referred to as The Higher Arithmetic or, more formally, Number Theory.

The work is presented in as informal manner as possible, merely looking at the numbers, observing and conjecturing without any proving.

There is an answer section for this work also together with some hints that might help in looking for relationships that are not obvious and, where suitable, suggestions are given for extending some of this work.

A Mathematics Formulary Study Guide (SG) ~ Notes ~ 5

Prime Numbers (SG ~ 14)

The building blocks.

Fibonacci Numbers (SG ~ 15)

How one of the simplest sequences to derive provides unexpected uses and relationships.

The Power of 2 (SG ~ 16)

An important sequence of growth that never fails to fascinate. How does it get to be so big?
The beginnings of logarithms.
A more general account of this can be found in the TRoL menu under Miscellaneous Papers.
It is intended that solutions are sought by noting the relevant relationships and using the tables for 2^n in the Formulary rather than engaging in any long (and boring) calculations.

Powers of N (SG ~ 17)

Looking at some of the identifying characteristics of numbers.

Factors (SG ~ 18)

An introduction to one of the topics of mathematics that continues to be a continual fascination and, as yet, an unsolvable problem – the factorisation of numbers.

More about formulas

Some additional thoughts and ideas about formulas can be found on the TRoL site under the heading Miscellaneous Papers where two relevant papers are listed**:**

BODMAS

&

A Formula Miscellany

Answers

can be found at

www.ex.ac.uk/trol/

=> A Mathematics Formulary

=> Answers

A **password** is needed to access the file, and this password is supplied to all those purchasing a copy of the Formulary.
As well as direct answers to most of the questions, there are also some suggestions for further work and Test Papers (with answers).

Familiarisation

The answers to all of these questions can be found by looking at the appropriate pages in the *Mathematics Formulary*.

Each question contains a mention of the heading under which the information is to be found and it is that heading which will need to be looked for in the List of Contents so as to locate the number of the correct page. As a help, in the earlier questions, the heading that is needed has been printed in **bold**.

1. What does the **symbol** ‰ mean?
2. Copy the formula for finding the area of a **trapezium**.
3. Give another name for a **torus**.
4. What is the particular name for a **polyhedron** having 20 faces?
5. Find the **Fibonacci Number** F_{20}
6. In the English alphabet the symbol Z is called 'zed' and is the 26th letter. What is its name and position in the **Greek Alphabet**?
7. What is the 'mole' used to measure in the **SI System of Units**?
8. Find another name for a **segment** of a **sphere**.
9. Give the value of **factorial** 12.
10. Write down the first term in the **series** for sin x
11. In working with a formula for sectors (of a circle), what does l stand for?
12. Give the A-factor for a regular polygon having 5 edges.
13. What is another name for the numbers given in the ${}^{n}\mathbf{C}_{r}$ table?
14. What conversion factor should be used to change pounds into kilograms?
15. How many faces does a parallelepiped have?
16. What is the meaning of the abbreviation RMS?
17. Find another expression for the term 'percentage profit'.
18. Write out the prime factors of 282.
19. Give the exact value of, and the abbreviation for, the prefix known as 'kibi'.
20. In the US System of Measurement, what is the exact size, in litres, of a gallon?
21. Among the formulas for cylinders, what does the symbol T represent?
22. What is 1 radian, in degrees, to 9 decimal places?
23. In temperatures, what is absolute zero on the Fahrenheit scale?
24. Find 2^{30}
25. How many escribed circles does a general triangle have?
26. On the 'standard' compass how many directions are named without using quarter-points?
27. What is the 21st prime number?
28. Find the 5th power of N when N = 45
29. In trigonometry, in which quadrant (other than the first) is the tangent positive?
30. Give the full name of, and symbol for, the correlation coefficient in statistics.

Conversions

It is often necessary to be able to change units of measurement in one system into those in another (like feet into inches). Of course it can only be done for units of the same type (such as length in the previous example). We cannot change units of length into units of volume (like feet into gallons!). There are various ways in which possible changes can be done and the following sections show how *The Formulary* can help in some cases.

In each section an example of the conversion is given, followed by a CheckOut for self-testing. The answers to the CheckOuts are given at the bottom but **not** in the same order as the questions are presented. Answers to the exercises are not given here.

Common Measures (page 41)

On this page can be found a set of conversion factors which allow some of the more commonly used units to be changed from one system to another.

Example 1: Given an area of 14 hectares which is needed to be known in acres, it can be seen that the conversion factor is × 2.471 and we can work out that

14 hectares is the same size as (14 × 2.471 =) 34.594 acres.

CheckOut 1 Try the following conversions.
4 feet into metres 17 acres to hectares 7 miles to kilometres 132 grams to ounces

Exercise 1: Do each of the following conversions.
(a) 6 inches to centimetres (b) 3 pounds to kilograms (c) 750 hectares to square miles
(d) 100 metres to yards (e) 10 gallons(UK) to litres (f) 10 gallons(US) to gallons(UK)

Temperatures (page 42)

On this page is a dual scale, showing both Celsius (°C) and Fahrenheit (°F) temperatures. This can be used to convert temperatures between these two systems with sufficient accuracy for most purposes. Note carefully how the scales are graduated. The °C scale is divided into 2's and the °F scale into 5's.

Example 2: Convert 160 °F into °C. We can see that 160°F corresponds approximately to 71°C.

CheckOut 2 Read off each of these temperature on the 'other'scale.
140°F 10°C 75°F 74°C

Exercise 2: Convert these temperatures in °F or °C into °C or °F as appropriate.
(a) 100°F (b) 46°C (c) 55°F (d) 96°C
(e) 70°F (f) 22°C (g) 10°F (h) ⁻2°C

Measures of Angle (page 11)

This page also contains a dual scale, but it is now circular rather than linear and shows how angles measured in either degrees or radians correspond. The graduations on the two scales are considerably different. On the outer scale marks are 2 degrees apart, while the divisions on the inner scale represent 0.05 radians.

Example 3: Change 152° into radians. We see that 152° on the outer scale closely matches 2.65 on the inner.

So 152° ≅ 2.65 rad (≅ means 'is approximately equivalent to')

CheckOut 3 Change these angles in degrees (or radians) into radians (or degrees).
86° 4.8 rad 195° 3.7 rad 43° 4.05 rad

Exercise 3: Convert each of the following angles as appropriate.
(a) 252° (b) 189° (c) 60° (d) 134°
(e) 0.4 rad (f) 1.1 rad (g) 3.35 rad (h) 5.58 rad

Answers

CheckOut 1:	11.263	1.2192	4.656	6.8799		
CheckOut 2:	50	165	60	24		
CheckOut 3:	212	275	232	1.5	0.75	3.4

ORDER

When working out $2+7-3+1$, say, the answer we get depends upon the order in which we do the different operations + and –
To see that we shall work it out using two different methods (or orders of working).

Method 1 (left to right) $2 + 7 - 3 + 1$
First do $2 + 7$ to get $9 - 3 + 1$
Next do $9 - 3$ to get $6 + 1$
Finally $6 + 1$ to get $= \mathbf{7}$

Method 2 (right to left) $2 + 7 - 3 + 1$
First do $3 + 1$ to get $2 + 7 - 4$
Next do $7 - 4$ to get $2 + 3$
Finally $2 + 3$ to get $= \mathbf{5}$

We have two different answers, clearly an undesirable and unacceptable state of affairs.
This leads to the first rule

WORK FROM LEFT TO RIGHT UNLESS TOLD OTHERWISE

The correct answer is **7**
The meaning of 'told otherwise' will become clear later, as we meet other rules.

That was using only + and − so now we bring in × (multiplication).
For this we shall work out $2+7\times3-1$ using four different methods.

Method 3 (+ × −) $2 + 7 \times 3 - 1$
First do $2 + 7$ to get $9 \times 3 - 1$
Next do 9×3 to get $27 - 1$
Finally $27 - 1$ to get $= \mathbf{26}$

Method 4 (+ − ×) $2 + 7 \times 3 - 1$
First do $2 + 7$ to get $9 \times 3 - 1$
Next do $3 - 1$ to get 9×2
Finally 9×2 to get $= \mathbf{18}$

Method 5 (− × +) $2 + 7 \times 3 - 1$
First do $3 - 1$ to get $2 + 7 \times 2$
Next do 7×2 to get $2 + 14$
Finally $2 + 14$ to get $= \mathbf{16}$

Method 6 (× + −) $2 + 7 \times 3 - 1$
First do 7×3 to get $2 + 21 - 1$
Next do $2 + 21$ to get $23 - 1$
Finally $23 - 1$ to get $= \mathbf{22}$

This time we have four different answers.
It might seem that the simple way to resolve this would be to always work from left to right, but that is not possible with our present mathematical notation, and we have another rule.

MULTIPLICATION MUST BE DONE BEFORE ADDITION AND SUBTRACTION

So *Method 6* is the correct way and the final answer is **22**
This is an instance of where we are 'told otherwise'.
Notice that after the × was done, the remaining + and – must be done from left to right.

Example 4: Evaluate $4.8 - 2.1 + 6.5 \times 1.3$
The × is a 'told otherwise' so we do that first: $6.5 \times 1.3 = 8.45$
This gives $4.8 - 2.1 + 8.45$ (which we work left to right) to get 11.15

CheckOut 4 Evaluate the following:

$5.8 + 1.7 - 2.3 + 4.75$	$2.1 + 1.3 \times 6.5 - 4.8$
$15.6 - 2.18 \times 4.75$	$27.8 + 3.4 - 2.6 \times 9$

Exercise 4: Evaluate each of the following:

(a) $2.38 + 7.3 \times 1.6 - 8.4$ (b) $5.4 - 5.4 \times 0.7$
(c) $6.8 \times 3.7 - 15.6 + 3.94$ (d) $4.81 \times 2.6 - 1.7 \times 3.6$
(e) $5.6 - 2.94 + 1.05 \times 4.47$ (f) $2.39 + 1.7 \times 2.8 \times 3.4$

Answers	CheckOut 4:	7.8	9.95	5.75	5.245

BRACKETS

After establishing the previous two rules we pause to wonder if we ever need to make an exception. For instance in $2 + 7 \times 3$ there might be a reason for wanting the $2 + 7$ to be done first rather than the 7×3 which the previous rules demand.
In fact it does happen and, so that our intentions can be made clear, we use

brackets.

These are always used in pairs and come in various shapes. The different varieties with their most commonly used descriptors are

( )	[ ]	{ }	< >
brackets	square brackets	curly brackets *or* braces	angle brackets

The most usual are (brackets), the rest have other more specialised uses.
The first one of the pair is known as the opening bracket, and the second as the closing bracket.
Now we need another rule to make our purpose clear.

CONTENTS OF BRACKETS MUST BE WORKED OUT FIRST

So, using the example at the top, if instead of $2 + 7 \times 3$ we write $(2 + 7) \times 3$
then we get $9 \times 3 = 27$. Without the brackets it is $2 + 21 = 23$

Example 5: Evaluate $9.6 + (3.2 - 1.5) \times 4.1$
First we must evaluate $(3.2 - 1.5)$ which makes it read $9.6 + 1.7 \times 4.1$
The order is then × followed by + to get **16.57**
Without the brackets it would have been 5.72

CheckOut 5 Evaluate the following:

$5.8 + (3.2 - 1.7) + 4.75$ $\qquad$ $(2.1 + 1.3) \times 6.5 - 4.8$
$2.18 \times (21.8 - 15.6)$ $\qquad$ $(9.6 + 2.18) \times (6.5 - 4.7)$

Exercise 5: Evaluate each of the following

(a) $(2.38 + 7.3) \times 1.6 - 8.4$ $\qquad$ (b) $15.4 - 1.6 \times (2.7 + 1.8)$
(c) $6.8 \times (15.6 - 3.7) + 3.94$ $\qquad$ (d) $4.81 \times 3.6 - (1.7 + 3.6)$
(e) $(5.6 - 2.94 + 1.05) \times 4.47$ $\qquad$ (f) $(2.39 + 1.7) \times 2.8 - 3.4$

Exercise 5A: Evaluate:

(a) $4.7 + 2.5 \times 1.9 - 5.6$ $\qquad$ (b) $6.5 - 1.3 + 4.1 \times 2.5$
(c) $(3.8 + 5.3) - 1.7 \times 2.9$ $\qquad$ (d) $(7.4 - 2.5) \times (1.2 + 3.8)$
(e) $17.4 - 2.4 + 3.8 + 5.12$ $\qquad$ (f) $17.4 - (2.4 + 3.8) + 5.12$
(g) $3 \times (8.2 - 4.7) - 7.09$ $\qquad$ (h) $2.6 + 9.37 - 1.65 \times 3.1$
(j) $14.3 + 2.2 \times 1.06 \times 5.8$ $\qquad$ (k) $7.3 \times 1.51 + 4.6 \times 2.8$

Answers CheckOut 5: 17.3 21.204 12.05 13.516

LINES

There are three different types of line to be aware of in this work and they all do something similar in the way that they group things together, rather like brackets.

The first type is shown in $\frac{7+5}{20\times3}$ and indicates a fraction. It means that the groups on the top and the bottom must be evaluated separately first, and then the top value is **divided** by the bottom value. This is the same as $(7 + 5) \div (20 \times 3)$

The second type is shown in $10/4$ and means the first value must be **divided** by the second value, so it is the same as $10 \div 4$ or as $(10 \div 4)$

> Note that if this second type of line (called a *solidus*) is used in the previous expression, it would need to be put in the form $(7 + 5)/(20 \times 3)$ and NOT $7 + 5/20 \times 3$ which is something much different.

The third type is shown in $\overline{7+2\times9}$ and is an old way of grouping things together. It is the same as using brackets and today we would write $(7 + 2 \times 9)$. However, this line (called a *vinculum*) still has one very particular use in today's mathematics where it is used to indicate that the root of an expression has to be found, without using brackets to make it clear.

We could write $\sqrt{}(7 + 2 \times 9)$ but prefer $\sqrt{}\,\overline{7+2\times9}$ which is usually written $\sqrt{7+2\times9}$

Evaluation of $\sqrt{7+2\times9}$ produces $\sqrt{25}$ which is 5

This leads to the observation that **LINES NEED TO BE TREATED LIKE BRACKETS**

Example 6: Evaluate $\sqrt{28.5-3.7\times4.2} - 1.7 + 13.92/1.6 \times 3.8 - \frac{5.4\times10.15}{1.6+1.9}$

After evaluating each of the three groups containing a line we have

$3.6 - 1.7 + 8.7 \times 3.8 - 15.66$

Then the multiplication, followed by working left to right gives **19.3**

CheckOut 6 Evaluate:

$24.3 - 18.72/7.8$	$\sqrt{35.72-3.1\times2.8}$
$7.46 + \frac{13.94+16.51}{2.1\times5.8}$	$\sqrt{\frac{8.8\times6.2}{13.4+20.7}+5.69}$

Exercise 6: Evaluate:

(a) $\sqrt{3.37+2.7\times4.1}$

(b) $15.7 - \frac{3.3\times5.4}{4.2-1.7}$

(c) $11.6 + 12.04/4.3 \times 5.9$

(d) $\frac{13.97+17.31}{12.4-7.8} + \sqrt{15.17+5.08}$

(e) $4.79 + \frac{30-8.108}{\sqrt{2.18+4.58}}$

(f) $\frac{873.925+181.883}{20.8\times3.6} - \sqrt{\frac{37.8\times9.61}{10.06-5.86}}$

Exercise 6A: Evaluate:

(a) $\sqrt{71.1+14.3\times16.2} - 5.6$

(b) $(7.2 - 4.8) \times \sqrt{2.1\times(12.5+6.4)}$

(c) $(5.78 + 13.12) \times \frac{22.13+17.6}{20.5-17.6}$

(d) $\sqrt{\frac{84.76+90.68}{51.7-38.8}+19.527/2.3}$

(e) $\sqrt{(21.3+11.8)\times(21.3-11.8)+24.11}$

(f) $(8.03 + 5.621)/8.03 + 5.621$

Answers	CheckOut 6:	5.2	2.7	9.96	21.9

LETTERS

So far we have only looked at work in which the numbers were already in place. However, this is of very limited use when a more general statement needs to be made which can be applied to a whole range of numbers, and this is most easily seen in the case of formulas.

Consider these instructions on how to find the length of the perimeter (P) of an oblong, when the lengths of its two different edges a and b are known. The formula is

$$\boldsymbol{P} = 2 \times (a + b)$$

From that, whenever we are given values for a and b we can find P

So when $a = 3.4$ and $b = 2.75$ we can write

$$\boldsymbol{P} = 2 \times (3.4 + 2.75)$$

The rules we have met previously say that what is in the brackets must be worked out before doing the multiplication. This leads to:

$$\boldsymbol{P} = 2 \times 6.15 = \mathbf{12.3}$$

The references and examples of formulas which follow are all to be found on the relevant pages of the *Formulary*.

Example 7: For a trapezium, find A when $a = 7.4$ $b = 9.8$ $p = 2.5$

Looking under trapezium we see that $\boldsymbol{A} = p \times (a + b) \div 2$

Re-writing this with the known values in place of the letters gives

$$\boldsymbol{A} = 2.5 \times (7.4 + 9.8) \div 2$$

(Treating brackets first)

$$= 2.5 \times 17.2 \div 2$$

(Working left to right)

$$= 43 \div 2$$ and so $\boldsymbol{A} = \mathbf{21.5}$

CheckOut 7

For a square: (i) find e when $P = 13.8$ (ii) find P when $A = 5.29$

For an oblong: (i) find P when $a = 5.7$ $b = 2.4$ (ii) find a when $P = 9.3$ $b = 1.87$

Exercise 7

(a) For a square: find P when $e = 7.6$
(b) For an oblong: find a when $A = 9.43$ $b = 2.3$
(c) For a parallelogram: find A when $p = 5.4$ $e = 3.7$
(d) For a triangle: find A when $b = 4.6$ $p = 3.9$
(e) For a square: find d when $A = 5.78$
(f) For an ellipse: find d when $a = 6.4$ $e = 0.8$
(g) For a frustum of a pyramid: find V when $A = 8$ $B = 2$ $h = 3.6$

Where needed the value used for π may be 3.14 or that of the π-key on the calculator.

Exercise 7A

(a) For a circle: find d when $C = 17.3$
(b) For a circle: find C when $A = 11.5$
(c) For a cylinder: find C when $V = 15.04$ $r = 4.7$
(d) For a cylinder: find T when $r = 2.6$ $h = 8.5$
(e) For a cone: find r when $V = 45$ $h = 7.3$
(f) For a cone: find s when $r = 3.4$ $l = 8.16$
(g) For a pyramid: find b when $V = 76.7$ $h = 9.4$
(h) For a sphere: find C when $A = 25.3$
(j) For a sector of a circle: find r when $A = 17.604$ $l = 3.26$

Answers	CheckOut 7:	16.2	9.2	3.45	2.78

CONTRACTIONS

It is strange that one of the most used operations in formulas is multiplication and yet the × sign is rarely seen. The reason for this is because the use of letters to represent numbers allows a contraction to be made that is not possible with numbers alone.

Consider a case where $a = 3.1$ $b = 2.64$ $P = 17.8$ $Q = 51.95$ and all have to be multiplied together. It could be written as $a \times b \times P \times Q$ which is clear enough, but it is usual to omit the × signs and simply write $abPQ$ only putting the signs back when needing to use the numbers:

$$3.1 \times 2.64 \times 17.8 \times 51.95$$

(Imagine the difficulty of working out what 3.12.6417.851.95 meant!)

Here we will refer to this idea of leaving out × signs as a **contraction**.

Another contraction is using a^2 to mean $a \times a$ and d^3 to mean $d \times d \times d$ and so on.

Then we can see that something like $4a^2brs^3V$ is a contraction for $4 \times a \times a \times b \times r \times s \times s \times s \times V$

Yet another contraction is when a letter or letters or a number is written next to a bracket like:

$a(x + y)$ $r^2h(l - r)$ $8(m + n)$ $(a + b)(x - y)$

meaning

$a \times (x + y)$ $r \times r \times h \times (l - r)$ $8 \times (m + n)$ $(a + b) \times (x - y)$

So we see **MULTIPLICATIONS HIDDEN IN CONTRACTIONS MUST BE IDENTIFIED**

Example 8: For a segment of a sphere, find V when $d = 9.4$ $h = 1.7$

Looking under segment of a sphere we see that $\boldsymbol{V} = \pi h^2(3d - 2h) \div 6$

meaning $\boldsymbol{V} = \pi \times h \times h \times (3 \times d - 2 \times h) \div 6$

putting in the values: $\boldsymbol{V} = \pi \times 1.7 \times 1.7 \times (3 \times 9.4 - 2 \times 1.7) \div 6$

(Dealing with the first two × signs and the two inside the brackets)

$$= 9.079 \times (28.2 - 3.4) \div 6$$

(Evaluating what is inside the brackets)

$$= 9.079 \times 24.8 \div 6$$

(Working left to right)

$$\boldsymbol{V} = \mathbf{37.53}$$

CheckOut 8

For a frustum of a cone: (i) find C when $D = 6.9$ $d = 3.5$ $l = 8.4$

(ii) find V when $D = 6.9$ $d = 3.5$ $h = 10.3$

Exercise 8

(a) For a zone of a sphere: find A when $d = 4.7$ $h = 1.2$

(b) For a torus: find S when $D = 18.4$ $d = 2.35$

(c) For a barrel: find V when $D = 1.7$ $d = 1.4$ $h = 2.3$

(d) For a sector of a circle: find $s°$ when $l = 21.3$ $r = 14.6$

(e) For a cuboid: find S when $a = 5.7$ $b = 4.6$ $c = 2.8$

(f) For an ellipse: find e when $a = 9.3$ $b = 7.4$

(g) For a sector of a sphere: find V when $d = 22.8$ $h = 3.1$

(h) For a cuboid: find c when $V = 420$ $a = 3.8$ $b = 7.7$

Answers CheckOut 8: 226.5 184.3 137.2 113.8

SUMMARY

of the rules and conventions governing the evaluation of formulas

CONTENTS OF BRACKETS MUST BE WORKED OUT FIRST

Remember that brackets always come in pairs.
Sometimes they are 'nested'. That is one or more pairs might be inside another.
In this case, find the innermost pair and work outwards.
For instance something like $a(b + 3(c - x))$ would need $(c - x)$ to be done, then multiplied by 3, then b added before, finally, multiplying by a.

LINES NEED TO BE TREATED LIKE BRACKETS

$\dfrac{3a^2 + b}{5c(a - d)}$ requires the top and bottom values to be found separately, and then the top value is divided by the bottom value.

$a + b/c - d$ means $b \div c$ must be done before working from left to right

$(a + b)/c - d$ means $(a + b)$ must be done before dividing by c

$(a + b)/(c - d)$ means $(a + b)$ must be done before dividing by $(c - d)$

$\surd$ requires the expression following (under an attached horizontal line) to be evaluated as one complete unit before being used by the rest of the formula.

MULTIPLICATIONS HIDDEN IN CONTRACTIONS MUST BE IDENTIFIED

Something like $4a^2bxs^3$ represents $4 \times a \times a \times b \times x \times s \times s \times s$
and $7a(p + q)(x - 2y)$ is shorthand for $7 \times a \times (p + q) \times (x - 2 \times y)$

MULTIPLICATION MUST BE DONE BEFORE ADDITION AND SUBTRACTION

and so also must division if any remains after all the above items have been cleared away.

In some (older) formulas an ‘of’ might be found. Replace it with ×

WORK FROM LEFT TO RIGHT UNLESS TOLD OTHERWISE

This will only apply when all

- **brackets** have been removed, and replaced by the value of their contents,
- **lines** have been dealt with as appropriate,
- **contractions** have either been written out in full or worked out,
- the operations left are all of the same priority;
  usually this will mean they are all either + and – **or** × and ÷

REGARDLESS of ALL the RULES

When constructing a formula, if there might be any possibility of a misunderstanding, always put in brackets. Too many is better than too few.

Formulas for Mensuration

Apart from being able to follow the instructions contained in a formula, put in the relevant numbers, and produce a final numerical solution, there are two other elements needed in practical work.

1. **Find the correct formula**. This means linking the descriptive language of what has to be found and what is given, to the letters (or symbols) by which it is identified, and then selecting the formula that connects those letters.
2. **Use the correct units**. This means not only working with the correct units in the formula, but also giving the answer in the units in which it is required.

Example 9: A pyramid has slant edges of length 17.3 cm and a perpendicular height of 14.6 cm. Find the length of its base edge.

Solution

Looking under pyramid we find 'slant edge' is represented by *s*,
'perpendicular height' by *h*, and 'base edge' by *b*.

It is good practice when identifying the letters to write them down with their values.

s = 17.3 cm h = 14.6 cm b = ?

Now we need a formula giving b in terms of s and h.

It is $\sqrt{2\times(s^2-h^2)}$ – note it could have been written as $\sqrt{2(s^2-h^2)}$

Evaluation, using the given values, produces 13.124786 and the final answer is:

The length of its base edge is 13.12 cm (to 2 decimal places)

Example 9a: An oil-drum is in the shape of a cylinder having a diameter of 40 cm and a height of 127 cm. Calculate its volume and then find how many litres of oil it will hold.

Solution

Following the procedure outlined above, under cylinder we find that we have

d = 40 cm, h = 127 cm, and $V = \pi \times d^2 \times h \div 4$

this leads to the first answer:

The volume of the container is 159 592 cu.cm (160 000 to 3 significant figures)

and, since there are 1 000 cu.cm in a litre, the final answer is:

The container will hold 160 litres of oil.

Exercise 9 (Give answers to an appropriate degree of accuracy.)

(a) A pyramid has base edges of length 9.4 cm and slant edges of length 12.7 cm. Find its perpendicular height.

(b) What is the volume of a torus which has an inside diameter of 15.2 cm and a cross-sectional diameter of 2.5 cm?

(c) Find the volume of a cone having a base diameter of 24.3 cm and a perpendicular height of 36.8 cm.

(d) The edges of a cuboid are of length 3.2 cm, 5.8 cm, and 10.4 cm. Find the length of its internal diagonal.

(e) What is the volume of a cube whose surface area is 86.64 sq.cm?

(f) Find the area of a sector which has been cut from a circle of radius 8.5 cm and has an angle of 75 degrees.

(g) The diagonal of a square is 10 cm long. What is its edge-length?

(h) A frustum of a cone has a base circle diameter of 34 cm, a top circle diameter of 19 cm, and a perpendicular height of 28 cm. What is its slant height?

(j) The shape in the preceding question was actually that of a bucket. What is its capacity in gallons?

(k) Calculate the surface area of a cube whose volume is 10.648 cu.cm.

Formulas for Trigonometry

All of these exercises concern triangles, and the symbols/letters used match those defined on the relevant pages of *A Mathematics Formulary*.
(Give all answers to an appropriate degree of accuracy.)

Exercise 10 **Right-angled Triangle**

1. Given $a = 3.4$ cm and $b = 5.8$ cm, find c.

2. Given $b = 5.1$ cm and $c = 9.3$ cm, find a.

3. Given $a = 4.7$ cm and $\angle A = 36°$, find c.

4. Given $b = 6.3$ cm and $\angle A = 43°$, find a.

5. Given $c = 12.2$ cm and $\angle A = 51°$, find $\angle B$.

6. Given $c = 8.5$ cm and $\angle B = 62°$, find a.

7. Given $b = 9.4$ cm and $c = 15.7$ cm, find $\angle B$.

8. Given $a = 7.6$ cm and $\angle B = 54°$, find b.

9. Given $a = 5.9$ cm and $c = 14.3$ cm, find b.

10. Given $b = 4.3$ cm and $\angle A = 48°$, find c.

11. Given $b = 7.1$ cm and $\angle B = 68°$, find c.

12. Given $a = 2.8$ cm and $c = 3.1$ cm, find $\angle B$.

Exercise 11 **General Triangle**

1. Given $a = 6.7$ cm, $b = 5.2$ cm, and $c = 3.8$ cm,
(a) find s,
(b) using the values of a, b, c, and s, find the area Δ,
(c) find the radius of the inscribed circle,
(d) find the radius of the circumscribed circle,
(e) find the radius of the escribed circle drawn on edge a,
(f) find the size of angle A.

2. Find the area of the triangle when $a = 3.6$ cm, $b = 7.4$ cm, and $c = 5.5$ cm.

3. Find the area of the triangle when $a = 18.3$ cm, $b = 15.6$ cm, and $C = 38°$.

4. Given $a = 12.4$ cm, $A = 47°$, and $B = 55°$, find b.

5. Given $a = 15.3$ cm, $b = 13.7$ cm, and $C = 49°$, find c.

6. Given $a = 23.1$ cm, $b = 34.5$ cm, and $c = 28.6$ cm, find A.

7. Given $a = 7.26$ cm, $b = 9.4$ cm, and $c = 10.3$ cm, find C.

8. Given $a = 14.8$ cm, $b = 12.3$ cm, and $c = 11.5$ cm, find R and r.

9. Find the area of the triangle when $a = 5.24$ cm, $b = 6.17$ cm, and $B = 63°$.

10. Given $a = 24.3$ cm, $c = 37.8$ cm, and $B = 126°$, find b.

11. Given $a = 8.47$ cm, $c = 10.2$ cm, and $C = 137°$, find A.

12. Given $a = 7.4$ cm, $b = 6.1$ cm, and $B = 42°$, find A and c.

Formulas for Poly-shapes

Many of the calculations needed in working with either polygons or polyhedrons (but only **regular** ones) can be handled by means of formulas, but in this case the formula has to be 'assembled' first.

Example 12: Find the area of a regular octagon whose edge length is 5 cm.

Solution

Looking under regular polygons we find 'edge length' is represented by $\boldsymbol{e}$ and

"**area** of the **polygon** is given by $A = \boldsymbol{e}^2 \times$ A-factor"

The table following gives the A-factor for an **octagon** as 4.8284

So the area of the regular octagon is $\boldsymbol{e}^2 \times 4.8284$ (usually written $4.8284\boldsymbol{e}^2$)

In this case $\boldsymbol{e} = 5$ and the final answer is:

The area of the octagon is 120.71 cm^2

Example 12a: A regular polygon having 30 edges, each 2.5 cm long, has to be drawn.
Find the radius of the circumcircle needed to make the drawing.

Solution

As before, we find the formula needed is $R = \boldsymbol{e} \times$ C-factor

But the C-factor for 30 edges is not given in the table.

Looking below the table we see C-factor = $\dfrac{1}{2\sin s^\circ}$ and $s^\circ = \dfrac{180^\circ}{n}$

Since $n = 30$ this leads to $s^\circ = 6^\circ$ and then $2 \times \sin 6^\circ = 0.2090569$

Dividing that into 1 gives a C-factor of 4.7833861

which has to be multiplied by $\boldsymbol{e}$ so the final answer is:

The radius of the circumcircle of the 30-gon is 11.96 cm (to 4 sig.figs.)

Exercise 12 (Give answers to an appropriate degree of accuracy) All named shapes are regular.

1. How many edges does a heptagon have?

2. What is the interior vertex angle of a nonagon?

3. Give the I-factor for a hexagon.

4. Calculate the area of a pentagon having 6.5 cm edges.

5. Find the incircle radius of a dodecagon with edges of length 3.7 cm.

6. An icosagon is a polygon having 20 edges (it is also known as a 20-gon).
What would be the area of one having all its edges of length 3.8 cm?

7. What is the V-factor for a tetrahedron?

8. Find the surface area of a tetrahedron with an edge length of 8.5 cm.

9. Calculate the volume of an octahedron whose edges are all 6.9 cm in length

10. A dodecahedron has edges 3.4 cm in length. What will be the radius of its insphere?

11. An icosahedron has 4.6 cm edges.
(a) What is its surface area?
(b) What is the radius of its circumsphere?
(c) How much bigger is the circumsphere (in volume) than the icosahedron?

12. Compare the volume of a cube with the volume of its circumsphere.

13. Find the perimeter of a decagon whose edges are each 5.8 cm long.

14. Compare the perimeter of a 100-gon with the circumference of its circumcircle.

Formulas for Professionals

Sheet SG ~ 12A will be needed to do these exercises.
(Give all answers to an appropriate degree of accuracy.)

Exercise 13

1. An observer standing on a cliff estimates he is about 300 feet above sea-level. What would be the distance of his observed horizon in miles?
2. A mother and child are standing on a beach at the water's edge. Their eye-levels are 1.6 and 1 metre respectively above sea-level.
   Looking seaward, how many miles further can the mother see than the child?
3. The navigator on the bridge of a ship knows his eye-level to be 17.4 metres above the sea. How many kilometres away will the navigator's horizon be?
4. The navigator (in the previous question) can just see the top of a lighthouse over the horizon. On the chart the lighthouse is marked as having a height of 34 metres.
   How far is the navigator from the lighthouse?
5. The wind of a hurricane can blow with a speed of 35 metres/second (which is nearly 80 m.p.h). Calculate the pressure generated by such a wind.
6. Find the force on the end of a house having an area of 120 m^2 when the wind (in the previous question) is blowing directly on it, given
   Force (in newtons) = Pressure (in bars) × Area (in m^2) × 100 000
7. Find the power transmitted by a shaft turning at 170 revolutions/minute under a torque of one million newton metres.
8. Analysis of a sample of one particular coal produced these results:
   Carbon 85%　　Hydrogen 6%　　Oxygen 5%
   Calculate the (theoretical) calorific value of that coal.
   (Note that the percentages do not add up to 100 since other things are present in the fuel which contribute little, or nothing, to the calorific value.)
9. Calculate the rate of discharge of water flowing through a triangular notch in a weir when the height of the water is measured as 17.5 mm.

Exercise 14

1. During a test for air-pollution, the technician got a reading of 1.3 on the reflectometer while using a clamp size of 25 mm. The volume was estimated to be 150 m^3.
   What was the smoke density?
2. A steel pipe of radius 37.5 mm has a wall-thickness of 6 mm.
   What will be its bursting pressure?
   For steel, Young's modulus is 210 000 MPa and Poisson's ratio is 0.29
3. During a hardness test on a piece of steel, the load was 1000 kg, the diameter of the ball was 25.4 mm and the diameter of the indentation was 4.25 mm.
   What was the hardness of that sample of steel as measured on the Brinell scale?
4. Calculate the value of K and the expected deflection of an aluminium beam given:
   Load is 1500 newtons; Modulus of elasticity is 70 000 newtons/mm^2;
   Length is 5000 mm; Moment of inertia is 520 000 mm^4; a is 0.1
5. An air-compressor has a discharge pressure of 40 bars and is pumping to a receiver whose pressure is 8 bars. For those conditions, $C = 0.03$ and $b = 0.01$
   Find the expected mass flow rate.

Formulas for Professionals

Distance of Observed Horizon

m = height (of eye) of observer in **metres**
f = height (of eye) of observer in **feet**
K = distance to horizon in **kilometres**
M = distance to horizon in **miles**

$$K = 3.56959\sqrt{m} \qquad K = 1.97073\sqrt{f}$$

$$M = 2.21804\sqrt{m} \qquad M = 1.22455\sqrt{f}$$

Discharge through a Triangular Notch

To measure the flow of water when it is running through a triangular notch cut in a weir.

D = discharge rate in **litres/minute**
H = height of water above bottom of notch in **mm**

$$D = \frac{\sqrt{H^5}}{360}$$

Flow Rate through a Pipe

d = diameter of pipe in **mm**
V = velocity of flow in **metres/sec**
F = rate of flow in **litres/minute**

$$F = \frac{15V\pi d^2}{1000}$$

Specific Energy of a Fuel

Also known as **Calorific Value**

C = % Carbon content
H = % Hydrogen content
O = % Oxygen content
S = Specific Energy in **kilojoules/kilogram**

$$S = 324C + 1442(H - \frac{O}{8})$$

Power transmitted by a Shaft

T = Torque in **newton metres**
R = Rate of turn in **revs/minute**
P = Power in **watts**

$$P = \frac{\pi TR}{22370}$$

Wind Pressure

V = Velocity of wind in **metres/sec**
P = Pressure in **bars**

$$P = 0.0016609V^2$$

Hardness of Material

P = load in **kilograms**
D = Diameter of ball in **mm**
d = diameter of indentation in **mm**
H = Hardness measured on the Brinell scale

$$H = \frac{P}{\frac{\pi D}{2}\left(D - \sqrt{D^2 - d^2}\right)}$$

Pendulum

L = Length of pendulum in **mm**
g = gravitational acceleration in **mm/sec²**
T = Time of one swing in **seconds**

$$T = \pi\sqrt{\frac{L}{g}} \qquad L = \frac{gT^2}{\pi^2}$$

Aerodynamics

S = Surface area under test in **m²**
C = Cross-sectional area of wind tunnel in **m²**
C_L = Coefficient of lift
C_D = Coefficient of drag

$$C_D = \frac{SC_L^2}{8C}$$

Air Pollution

S = Smoke density in **μg/m³** (micrograms/m³)
R = Reflectometer reading
C = Clamp size in **mm**
V = Volume of air in **m³**

$$S = 1.39C(91672 - 3332R + 49.62R^2 - 0.3533R^3 + 0.000986R^4) \div V$$

Bursting Pressure of a Cylindrical Pipe

p = pressure to burst pipe in **MPa** (megapascals)
E = Young's modulus for material of pipe in **MPa**
μ = Poisson's ratio for material of pipe
R = Radius of pipe in **mm**
t = thickness of pipe in **mm**

$$p = \frac{Et^3}{4(1-\mu^2)R^3}$$

Deflection of a Beam

d = deflection (maximum) of beam in **mm**
W = Load on beam in **newtons**
E = Modulus of elasticity of beam in **newtons/mm²**
I = Moment of inertia of beam-section in **mm⁴**
L = Length between supports in **mm**
a = Fraction of length of beam not loaded
$K = 5 - 24a^2 + 16a^4$

$$d = \frac{WL^3K}{584EI(1-2a)}$$

Compressed Air Flow

M = Mass flow in **kg/sec**
P_0 = Pressure of discharge in **bars**
P_1 = Pressure in receiver in **bars** ($P_0 > P_1$)
C = Constant determined from experiments
b = value taken from appropriate tables

$$M = P_0 C\sqrt{1 - \left(\frac{P_1/P_0 - b}{1-b}\right)^2}$$

Formulary Cross Puzzle

The grid is to be filled in, using the clues provided below, with the value of **?** which has to be found.

Clues are provided in the briefest possible form and, where letters/symbols are given, they are the same as those in the *Mathematics Formulary*.

Where necessary, answers have to be truncated (with correct rounding) to fit the grid.

Unnecessary zeros are not written. For example: 0.46 is entered as .46

A decimal point occupies a single cell.

One entry has been made.

1		2	3	■	4	■	5	6		7
	■	8							■	
	■	■		■		■		■	■	
9	10	■	11	12		13		■	14	
■	15	16	■	17			■	18		■
19					■	20				
■	21		■	22	23		■	24		■
25 **3**		■	26				27	■	28	29
•	■	■		■		■		■	■	
1	■	30						31	■	
32 **4**				■		■	33			

Across

1. Square: edge = 2.8
Area = ?

5. Oblong: area = 450
a = 36 b = ?

8. Cube: S = 37
Volume = ?

9. Cylinder: C = 14.8
r = 7 Volume = ?

11. F_{25} = ?

14. Cuboid: a = 16.3
b = 21.5 c = 33.4 d = ?

15. Sphere: r = 0.4
Volume = ?

17. Torus: D = 8.2
d = 5 S = ?

18. Sector of a circle:
A = 102 l = 12 r = ?

19. Barrel: h = 12.7
D = 10.8 d = 7.2 V = ?

20. Segment of a circle:
r = 47.7 s = 86 A = ?

21. Cube: Volume = 4.7
S = ?

22. Tetrahedron:
V-factor = ?

24. Ellipse: a = 4.5
b = 3.4 A = ?

25. Segment of a circle:
r = 27 h = 8 c = ?

26. ${}^{18}C_8$ = ?

28. Cone: l = 77
h = 51 r = ?

30. 4486 miles = ? km

32. Pyramid: b = 4.7
l = 5 h = ?

33. 99-gon: C-factor = ?

Down

1. Circle: area = 430
Circumference = ?

2. Sphere: d = 5.36
Volume = ?

3. Ellipse: a = 30.47
e = 0.75 d = ?

4. Cube: V = 12246
e = ?

5. Cone: d = 20.8
h = 11.7 Volume = ?

6. Pyramid: b = 13
s = 27 h = ?

7. 1 radian = ? degrees

10. e = ?

12. 569 270 MW = ? GW

13. Dodecahedron: e = 32.289
surface area = ?

14. Sphere: V = 57364
d = ?

16. Frustum of cone: d = 126
D = 276 h = 308 l = ?

18. ESE by S = ? degrees

23. 26^3 = ?

25. π = ? ✓

26. Torus: D = 4.06
d = 0.605 V = ?

27. Segment of a circle:
s = 53.7 r = 82.61
h = ?

29. Segment of a sphere:
h = 1.94 d = 15.8
V = ?

30. 165° F = ?° C

31. Frustum of a pyramid:
A = 7.3 B = 9.9
h = 8.75 V = ?

Prime Numbers

If the natural numbers (1, 2, 3, 4, 5, ...) are written out in order and every prime number is then marked in some way, perhaps like this

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28

it is soon clear that they are very unevenly scattered, and this can be confirmed by looking at the tables on pages 30 to 33. This observation leads to all sorts of questions and conjectures, and some starting points are suggested here.

1. What is the largest gap to be found between two consecutive prime numbers up to 360?

2. What is the largest gap to be found between two consecutive prime numbers up to 7000?

3. Christian Goldbach (1690-1764) conjectured that every even number (greater than 2) can be written as the sum of **two** prime numbers. For example 26 = 3 + 23.

Show that this is true for the even numbers from 10 to 30.

Note that this conjecture has not yet been proved to be true for **all** even numbers.

4. Another conjecture is that all prime numbers greater than 3 are either 1 less, or 1 more, than a multiple of 6. For example: 17 = 18 – 1 and 19 = 18 + 1.

Show that this is true for all the prime numbers up to 100.

Does it then follow, that any multiple of 6 must have a prime number either preceding it or following it?

5. Every even number can be written as the difference of two primes in more than one way. For example: 4 = 11 – 7 or 17 – 3 or 23 – 19 and so on.

Using **only the first 25 primes** (that is all those less than 100) –

Find all the ways of writing 4 as the difference of two primes.

How many ways are there of writing 6 as the difference of two primes?

What is the smallest even number that cannot be written under these rules?

6. The prime numbers 107, 113, 149, are of some interest since, even when written in reverse order, 701, 311, 941, they are still prime numbers. These are reversible primes.

Find some other reversible primes.

7. The set of digits 1, 1, and 3, are of interest in relation to prime numbers because, no matter in what order they are written 113, 131, or 311, they are always prime.

Find some other sets of digits with that property.

8. What is the next prime number after 6997 ?

The **smallest** prime number is **2**

The largest known prime number

(in November 2003) is

$2^{13\,466\,917} - 1$

Written out in full it would be

4 053 946

digits long.

Make an estimate of how much time it would take to write it out in full, or how much space it would need.

Fibonacci Numbers

1. Consider three consecutive Fibonacci numbers, say ... 3 5 8 ...
We see that 3×8 (first times the last) is equal to $5^2 - 1$ (middle squared take 1)
Is this always true? What about another three say, ... 5 8 13 ...
Now we have $5 \times 13 = 8^2 + 1$ and the sign is different.
What about other sets of three? Are there any rules that cover the various possibilities?

2. Consider four consecutive Fibonacci numbers, say ... 3 5 8 13 ...
We see that $5 \times 8 - 3 \times 13 = 1$
Is that always true for sets of four consecutive Fibonacci numbers?

3. Take two consecutive Fibonacci numbers, say ... 5 8 ...
and note that $5^2 + 8^2 = 89$ which is also a Fibonacci number.
What is the connection between 5, 8, and 89?

4. Writing O for odd and E for even, the Fibonacci sequence starts off
O O E O O E ...
Does it continue like that? If so, why?
Will the 958th number be odd or even?

5. Looking at only the **last** digits of the Fibonacci numbers we read
1 1 2 3 5 8 3 1 4 5 9 4 3 7 0 7 7 ...
How long is it before a pattern becomes apparent and the cycle is repeated?
Is there any pattern in the **first** digits?

6. Which of the Fibonacci numbers are divisible by 5?
and which by 2?
and which by 3?
and which by 4?

7. Take the first *four* numbers of the Fibonacci sequence and add them. The total is 7
For the first *five* numbers it is 12
For the first *six* numbers it is 20
For the first *seven* numbers it is 33
What is the connection between the *amount* of numbers being added, and the total?
Then, what will be the total of the first *fifty* numbers?

8. Start with the first number of the Fibonacci sequence, take *every other* number and list them to make a new sequence like this
1 2 5 13 34 89 233 610 ...
Now, add the first three, the first four, the first five, and so on.
The answers are all Fibonacci numbers, but which ones?
Start with the second number and take every other number, what happens then?

9. Choose any two *consecutive* numbers of the Fibonacci sequence and divide the larger one of the two by the smaller. Do this for several different pairs starting with $3 \div 2$
Can any conclusions be drawn from the results?

10. Choose any Fibonacci number (other than 1), then find a second Fibonacci number into which the first will divide exactly. Find a rule that links the two numbers.
List at least three other Fibonacci numbers into which the Fibonacci number nearest to half-a-million goes exactly.

The Power of 2

1. There is an old story that the person who invented chess asked to be rewarded with a gift of corn, and that the quantity of corn was to be decided by using the 64 squares of the chess-board itself, in this way:

1 grain for the 1st square
2 grains for the 2nd square
4 grains for the 3rd square
8 grains for the 4th square
16 grains for the 5th square
and so on …

So, simply doubling the quantity to be given for each square means that, by the 5th square, the reward would have totalled 31 grains of corn. That's not many so far, but there is still a long way to go to cover all 64 squares.

Exactly how many grains of corn was the inventor asking for?

2. A piece of paper is torn in half and one piece is placed on top of the other to make a small pile. This pile (of 2 sheets) is then torn in half and the results of that tearing are placed in a new pile.
This is continued, tearing each pile in half and making a new pile with all the pieces, until a total of 50 'tearings' have been done. Assuming 100 sheets of the paper have a thickness of 10 mm (or 1 cm) …

… what will be the height of the final pile?

3. Kim has a secret. Kim shares this secret with two friends. Each of these shares Kim's secret with another two friends who each share it with another two friends, and so on. Assuming that this pattern is continued, that each sharing takes one hour to happen, and that no one receives the secret twice …

… how long will it be before the entire world knows Kim's secret?

(Assume the population of the world to be about 6.5 billion.)

4. Access to a table giving exact values of 2^n allows us to do accurate multiplication very quickly. It is based upon one of the laws of indices which states that

$$a^m \times a^n = a^{m+n}$$

Suppose we wish to work out 2048×65536
Reference to a table of 2^n allows this to be re-written as

$$2^{11} \times 2^{16}$$

The law quoted above then gives the answer as

$$2^{11+16} \text{ or } 2^{27}$$

which the same table shows to be 134 217 728

Find the exact values of

(a) 8 388 608 × 134 217 728 (b) 1 073 741 824 × 549 755 813 888

Numbers and their Digits

1. The **last digit** of any number can obviously be anything from 0 to 9. But when we look at the last digits of the different powers of N we can notice two things. One is the regularity of the order in which the various digits appear, and the other is that some digits do not appear at all.

> Which four digits **never** appear at the end of N^2 ?
> What power of N has only four possibilities for its last digit?
> For which power of N does the last digit match that of N itself ?
> If the last digit of N^3 is 3, what do you know about the last digit of N ?

2. The **digital root** of a positive whole number is found by adding together all its digits to make a new number, then adding the digits of that, and so on, until only a single digit remains. That single digit is known as the digital root of the number.
Example: $8579 \rightarrow 8 + 5 + 7 + 9 = 29 \rightarrow 2 + 9 = 11 \rightarrow 1 + 1 = 2$
So the digital root of 8579 is 2
Note that the digital root will always be a single digit from 1 to 9

> List the digital roots of N^2 for various values of N, continuing the list until a clear pattern can be seen.
> Do the same for N^4 and then work out the digital root of 137^4
> Find a power of N which has a digital root of 6

3. Investigate the sequence of remainders when N^2 is divided by 8.

4. Is $N^3 - N$ always exactly divisible by 3?
Investigate a similar possibility for $N^5 - N$ and $N^7 - N$.

5. Looking at a table of **factorials** we see that from 5! onwards they all end in zero. More than that, the number of zeros at the end is steadily increasing so that 40! ends in 9 zeros.
At the end of 100! how many zeros must there be?

Factors

A **factor** is a number which divides exactly into another number.

(Note here we are only talking about whole numbers.)

Examples: 1 is a factor of 5   3 is a factor of 6   4 is a factor of 12

7 is a factor of 7   2 and 17 are factors of 68   and so on …

Every number (with the exception of 1) must have at least two factors: 1 and the number itself.

This leads to the definition of a **prime number** being
a number which has two, and only two, factors.

Mathematicians have always been interested in numbers and their factorisation, and part of this interest has been concerned with how many factors there are in any number. For instance, here are the numbers 20 to 40 and underneath each is the number of factors each one has:

20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
6	4	4	2	8	3	4	4	6	2	8	2	6	4	4	4	7	2	4	4	8

We know what it means when a number has only two factors but, noticing that nearly all the numbers have an *even* number of factors, what does it mean when a number has an *odd* number of factors?

A **prime factor** is a factor which is itself prime.

Example: 1, 2, 3, 4, 6, 12 are all factors of 12
but the only prime factors of 12 are 2 and 3

There is considerable interest in prime factorisation since it shows just how a number is made up.

For instance, knowing that $12 = 2 \times 2 \times 3$ (usually written $2^2 \times 3$) shows that both 4 ($= 2 \times 2$) and 6 ($= 2 \times 3$) must be factors of 12 and that, together with 1, 12 and 2, 3, there can be no other factors of 12

In short, 12 has 3 prime factors but 6 factors in total.

(Trivial with small numbers, but a very important tool when working with large numbers.)

So, is there any connection between how many prime factors are needed to make a number, and how many factors that number has altogether?

For instance, 30 has 3 prime factors ($2 \times 3 \times 5$) and 8 factors in total.
40 has 4 prime factors ($2 \times 2 \times 2 \times 5$) and 8 factors in total.

How many factors, in total, will each of these numbers have?

1. 79 235 ($= 5 \times 13 \times 23 \times 53$)   **2.** 24 156 ($= 2^2 \times 3^2 \times 11 \times 61$)

3. 3960 ($= 2^3 \times 3^2 \times 5 \times 11$)   **4.** 2592 ($= 2^5 \times 3^4$)

Work out, and list in size order (smallest to largest), all the factors for each.